$$\mathcal{L}\{\cos bt\} = \int_0^\infty e^{-st}\cos bt\,dt = \lim_{N\to\infty}\int_0^N e^{-st}\cos bt\,dt$$

$$= \lim_{N\to\infty}\left(-\frac{s}{s^2+b^2}e^{-st}\cos bt - \frac{b}{s^2+b^2}e^{-st}\sin bt\right)\Big|_0^N$$

$$= \lim_{N\to\infty}\left(-\frac{s}{s^2+b^2}\left(e^{-sN}\cos bN - 1\right) - \frac{b}{s^2+b^2}\left(e^{-sN}\sin bN\right)\right)$$

$$= -\frac{s}{s^2+b^2}(0-1) - \frac{b}{s^2+b^2}(0) = \frac{s}{s^2+b^2}$$

u		dv
$\cos bt$	$+$	e^{-st}
$-b\sin bt$	$-$	$-\frac{1}{s}e^{-st}$
$-b^2\cos bt$	$+$	$\frac{1}{s^2}e^{-st}$

$$\int e^{-st}\cos bt\,dt = -\frac{1}{s}e^{-st}\cos bt + \frac{b}{s^2}e^{-st}\sin bt - \int \frac{b^2}{s^2}e^{-st}\cos bt\,dt$$

$$\left(1+\frac{b^2}{s^2}\right)\int e^{-st}\cos bt\,dt = -\frac{1}{s}e^{-st}\cos bt + \frac{b}{s^2}e^{-st}\sin bt$$

$$\downarrow$$

$$= \frac{s^2+b^2}{s^2}$$

$$\int e^{-st}\cos bt\,dt = -\frac{s}{s^2+b^2}e^{-st}\cos bt - \frac{b}{s^2+b^2}e^{-st}\sin bt$$

$\lim_{N\to\infty} e^{-sN}\cos bN = 0$ (BY SQUEEZE TH'M)

$$-1 \le \cos bN \le 1$$

$$-e^{-sN} \le e^{-sN}\cos bN \le e^{-sN}$$

$$\lim_{N\to\infty} -e^{-sN} = \lim_{N\to\infty} e^{-sN} = 0 \quad (s>0)$$

$\lim_{N\to\infty} e^{-sN}\sin bN = 0$ (BY SQUEEZE TH'M)

$$-1 \le \sin bN \le 1$$

$$-e^{-sN} \le e^{-sN}\sin bN \le e^{-sN}$$

$$\lim_{N\to\infty} -e^{-sN} = \lim_{N\to\infty} e^{-sN} = 0$$

① $\mathcal{L}$ IS LINEAR

② $\mathcal{L}\{1\} = \frac{1}{s}$ ⑧ $\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$ IF...

③ $\mathcal{L}\{t\} = \frac{1}{s^2}$ ⑨ $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} \mathcal{L}\{f(t)\}$

④ $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ ⑩ $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

⑤ $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ IF $\mathcal{L}\{f\} = F(s)$

⑥ $\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}$ ⑦ $\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$

FIND $\mathcal{L}\{e^{-3t}\cos 2t\}$

$$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$$

$$\mathcal{L}\{e^{-3t}\cos 2t\} = \frac{s+3}{(s+3)^2+4}$$

IF $\mathcal{L}\{f\} = F(s)$ AND f IS CONT. ON $[0,\infty)$ AND f' IS PIECEWISE CONT. ON $[0,\infty)$ AND f IS OF EXPONENTIAL ORDER

$$\mathcal{L}\{f'\} = \int_0^\infty e^{-st} f'(t)\,dt = \lim_{N\to\infty} e^{-st} f(t)\Big|_0^N + \int_0^\infty s e^{-st} f(t)\,dt$$

u		dv
e^{-st}	$+$	$f'(t)$
$-se^{-st}$	$-$	$f(t)$

$$= \lim_{N\to\infty} \left(e^{-sN} f(N) - f(0)\right) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= \lim_{N\to\infty} e^{-sN} f(N) + s\mathcal{L}\{f\} - f(0) = s\mathcal{L}\{f\} - f($$

$$\int e^{-st} f'(t)\,dt = e^{-st} f(t) + \int s e^{-st} f(t)\,dt$$

$=0$ SINCE f IS OF EXPONENTIAL ORDER

SANITY CHECK! $\mathcal{L}\{t'\} = \mathcal{L}\{1\} = \frac{1}{s}$

$s\mathcal{L}\{t\} - 0 = s \cdot \frac{1}{s^2} = \frac{1}{s}$ ✓

$$\mathcal{L}\{(\sin bt)'\} = s\,\mathcal{L}\{\sin bt\} - \cancel{\sin b(0)}$$

$$\mathcal{L}\{b\cos bt\} = s\,\mathcal{L}\{\sin bt\}$$

$$\frac{b}{s}\mathcal{L}\{\cos bt\} = \mathcal{L}\{\sin bt\} = \frac{b}{s}\,\frac{s}{s^2+b^2} = \frac{b}{s^2+b^2}$$

$$\mathcal{L}\{t\,f(t)\} = \int_0^\infty e^{-st}\,t f(t)\,dt \quad ?$$

IF $F(s) = \mathcal{L}\{f(t)\}$

$$\frac{dF}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty \frac{d}{ds}\left(e^{-st} f(t)\right) dt$$

$$= \int_0^\infty -te^{-st} f(t)\,dt = -\int_0^\infty e^{-st}\,t f(t)\,dt$$

$$F'(s) = \frac{dF}{ds} = -\mathcal{L}\{t f(t)\}$$

$$\mathcal{L}\{t f(t)\} = -F'(s) = -\frac{d}{ds}\mathcal{L}\{f\}$$

SANITY CHECK IF $f = e^{at}$

SANITY CHECK: $\mathcal{L}\{t \cdot 1\} = \mathcal{L}\{t\} = \frac{1}{s^2}$

$$-\frac{d}{ds}\mathcal{L}\{1\} = -\frac{d}{ds}\frac{1}{s} = -\left(-s^{-2}\right) = s^{-2} = \frac{1}{s^2}$$

$$\mathcal{L}\{t\,f(t)\} = -\frac{d}{ds}\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{t^2 f(t)\} = \mathcal{L}\{t \cdot t f(t)\} = -\frac{d}{ds}\mathcal{L}\{t f(t)\}$$

$$= -\frac{d}{ds}\left(-\frac{d}{ds}\mathcal{L}\{f\}\right)$$

$$= \frac{d^2}{ds^2}\mathcal{L}\{f\}$$

$$\mathcal{L}\{t^n f(t)\} = -\frac{d}{ds}\left(-\frac{d}{ds}\left(-\frac{d}{ds} \cdots \mathcal{L}\{f\}\right)\right)\cdots)$$

$$= (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f\}$$

$$\mathcal{L}\{t^2\} = (-1)^2\frac{d^2}{ds^2}\mathcal{L}\{1\} = \frac{d^2}{ds^2}\frac{1}{s} = \frac{d^2}{ds^2}s^{-1} = \frac{d}{ds}(-s^{-2}) = 2s^{-3} = \frac{2}{s^3} = \frac{2!}{s^3}$$

$$\mathcal{L}\{t^3\} = -\frac{d}{ds}\mathcal{L}\{t^2\} = -\frac{d}{ds}2s^{-3} = 6s^{-4} = \frac{6}{s^4} = \frac{3!}{s^4}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

SANITY CHECK: $\mathcal{L}\{t^1\} = \frac{1!}{s^{1+1}} = \frac{1}{s^2}$ ✓